# 8

# Linear constitutive equations

## 8.1 Constitutive equations and ideal materials

The results given so far in this book apply equally to all materials. In themselves they are insufficient to describe the mechanical behaviour of any particular material.

To complete the specification of the mechanical properties of a material we require additional equations, which are called *constitutive equations*. These are equations which are particular to individual materials, or classes of materials, and they serve to distinguish one material from another. The mechanical constitutive equation of a material specifies the dependence of the stress in a body on kinematic variables such as a strain tensor or the rate-of-deformation tensor. Normally thermodynamic variables, especially temperature, will also be involved, but we shall make only brief references to these. Constitutive equations are also required in other branches of continuum physics, such as continuum thermodynamics and continuum electrodynamics, but these problems are outside the scope of this book, and we shall only discuss constitutive equations for the stress.

The mechanical behaviour of real materials is very diverse and complex and it would be impossible, even if it were desirable, to formulate equations which are capable of determining the stress in a body under all circumstances. Rather, we seek to establish equations which describe the most important features of the behaviour of a material in a given situation. Such equations can be regarded as defining *ideal materials*. It is unlikely that any real material will conform exactly to any such mathematical model, but if the ideal material is well chosen its behaviour may give an excellent approximation to that of the real material which it models. The model should be selected with the application as well as the material in mind, and the same real material may be represented by different ideal materials in different circumstances. For example, the theory of incompressible viscous fluids gives an excellent description of the behaviour of water flowing through pipes, but is useless for the study of the propagation of sound waves through water, because for sound-wave propagation a model which takes into account the compressibility of water is essential.

Historically, the constitutive equations which define the classical ideal materials (linear elastic solids, Newtonian viscous fluids, etc.) have been developed separately. In applications of these theories this separation is natural. However, at the formulative stage there are advantages in a unified approach which clarifies relations between the different special theories. Also it is possible to formulate some general principles which should be followed in the construction of constitutive equations.

A first requirement which any constitutive equation must satisfy is that of dimensional homogeneity: the dimensions of all terms in a constitutive equation must be the same. Since a constitutive equation always includes constants or functions which characterize the material under consideration, and these quantities have dimensions, the dimensional homogeneity requirement is usually not difficult to satisfy.

Constitutive equations should not depend on the choice of the coordinate system (although they may be expressed in terms of components relative to any selected coordinate system). They therefore take the form of relations between scalars, vectors and tensors.

An important restriction on mechanical constitutive equations is the requirement that the stress response of a body to a deformation is not affected by rigid-body motions, so that the stress in a body depends only on the change of shape of the body and is not affected (except for the change in orientation of the stress field relative to fixed axes) by a superposed motion in which the body moves as a whole. To formalize this requirement we specify that if a body undergoes two time-dependent motions, which differ from each other by a time-dependent rigid-body motion, then the same stress results from each of these motions. This is essentially equivalent to saying that constitutive equations are invariant under translations and rotations of the frame of reference; two observers, even if they are in relative motion, will observe the same stress in a given body.

Materials are usually regarded as either solids or fluids, and fluids are subdivided into liquids and gases. We do not attempt a precise definition of this classification; the dividing lines are not always clear and there are materials which possess both solid-like and fluid-like properties. The characteristic property of a fluid is that it cannot support a shearing stress indefinitely, so that if a shearing stress is applied to a body of fluid and maintained, the fluid will flow and continue to do so as long as the stress remains. A solid, on the other hand, can be in equilibrium under a shear stress. Some solids possess a natural configuration which they adopt in a stress-free state and to which they eventually return if a stress is imposed and then removed; if a natural configuration exists it is usually convenient, though not essential, to adopt it as the reference configuration. Fluids have no natural configuration and, given sufficient time, will adapt to the shape of any container in which they are placed.

## 8.2 Material symmetry

Most materials possess some form of *material symmetry*. The commonest case is that in which the material is *isotropic*; an isotropic material possesses no preferred direction and its properties are the same in all directions. It is impossible to detect the orientation in space of a sphere of isotropic material by performing an experiment on it. Many real materials are isotropic or nearly so; these include common fluids like air and water, metals in their usual polycrystalline form, concrete, sand in bulk, and so on. Other common materials have strong directional properties; an example is wood, whose properties along its grain are quite different from the properties across the grain. Single crystals of crystalline materials have directional properties which arise because their atoms are arranged in regular patterns, and this gives rise to the various classes of *crystal symmetry*. A material which possesses a single preferred direction at every point is said to be *transversely isotropic*. An example of such a material is a composite material which consists of a matrix reinforced by fibres arranged in parallel straight lines. Over length scales which are large compared to the fibre diameters and spacings, such a material may be regarded as macroscopically homogeneous, and the fibres introduce a preferred direction which is a characteristic of the composite material.

We consider material symmetries of two types; rotational and reflectional.

*Rotational symmetry.* Suppose a spherical volume element undergoes the homogeneous deformation illustrated in Fig. 8.1. A typical particle initially at $P_0$ moves to $P_1$ and the deformation is described by the equations

$$\boldsymbol{x} = \boldsymbol{F} \cdot \boldsymbol{X} \tag{8.1}$$

where, since the deformation is homogeneous, the components $F_{iR}$ of $\boldsymbol{F}$ depend only on $t$.

Now suppose that the element undergoes a second deformation, which is similar to the first except that the entire deformation field (but not the body) is rotated through an angle $\alpha$ about an axis $\boldsymbol{n}$. Thus if $\boldsymbol{Q}$ is the tensor defined by (6.11), the particle which is initially at $\boldsymbol{Q} \cdot \boldsymbol{X}$ moves in the second deformation to the point $\boldsymbol{Q} \cdot \boldsymbol{x}$, where

$$\boldsymbol{Q} \cdot \boldsymbol{x} = \boldsymbol{F} \cdot \boldsymbol{Q} \cdot \boldsymbol{X} \tag{8.2}$$

The second deformation is illustrated for the case in which $\boldsymbol{n} = \boldsymbol{e}_3$ in Fig. 8.1(c); in it the particle initially at $Q_0$ moves to $Q_2$, where

$$\angle P_0 O Q_0 = \angle P_1 O Q_2 = \alpha$$

The deformed sphere has the same shape in the two configurations, but the second is not derived from the first by a rigid rotation. Although the two deformations (8.1) and (8.2) are related, they are distinct, and in the absence of appropriate material symmetry they will give rise to different stress responses. For example, the forces which accompany a given extension in the direction $OP_0$ will be different from those associated with the same extension in the direction $OQ_0$. However, for a given material it may happen that for certain rotations the result of rotating the deformation field through the rotation defined by $\boldsymbol{Q}$ is to

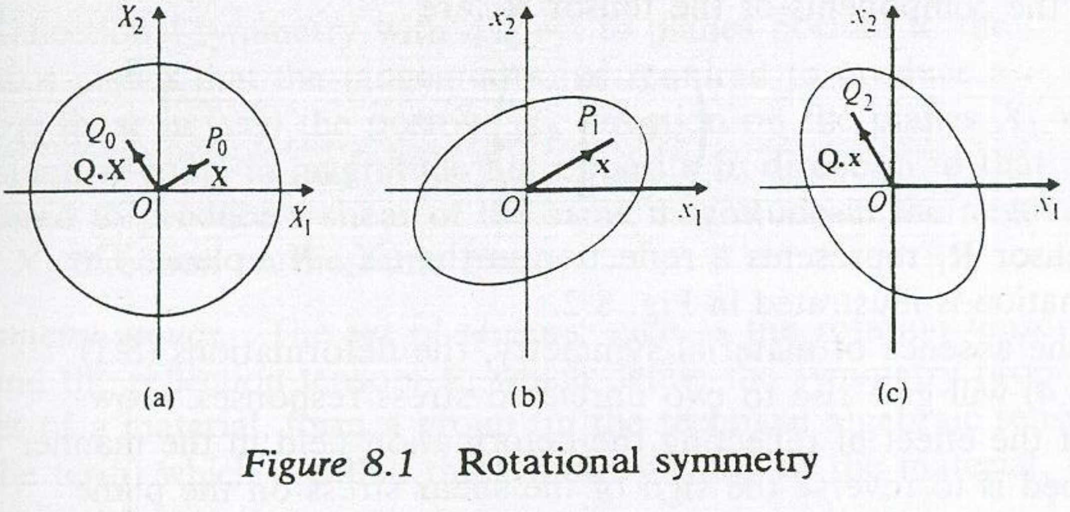

*Figure 8.1* Rotational symmetry

produce the same rotation of the stress field. In this case, if the deformation (8.1) gives rise to a stress tensor $\boldsymbol{T}$, then the deformation (8.2) gives rise to a stress tensor $\boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{T} \cdot \boldsymbol{Q}$. We then say that the material has *material symmetry* (relative to the specified reference configuration) for the rotation determined by $\boldsymbol{Q}$.

As a simple example, the tensor $\boldsymbol{Q}$ with components $Q_{iR}$, where

$$(Q_{iR}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents an anti-clockwise rotation of magnitude $\frac{1}{2}\pi$ about the $X_3$-axis. If the material has rotational symmetry for this rotation, then the force $\boldsymbol{p}_1$ required to produce a given extension in the $X_1$ direction has the same magnitude as the force $\boldsymbol{p}_2$ required to produce the same extension in the $X_2$ direction.

*Reflectional symmetry.* Now consider a further homogeneous deformation of the spherical volume element which is the mirror image of the deformation (8.1), in some plane which for definiteness we take to be the plane $X_1 = 0$. This deformation is defined by

$$\begin{pmatrix} -x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \begin{pmatrix} -X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{8.3}$$

or

$$\boldsymbol{R}_1 \cdot \boldsymbol{x} = \boldsymbol{F} \cdot \boldsymbol{R}_1 \cdot \boldsymbol{X} \tag{8.4}$$

where the components of the tensor $\boldsymbol{R}_1$ are

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.5}$$

The tensor $\boldsymbol{R}_1$ represents a reflection in the $(X_2, X_3)$ plane. The deformation is illustrated in Fig. 8.2.

In the absence of material symmetry, the deformations (8.1) and (8.4) will give rise to two unrelated stress responses. However, if the effect of reflecting the deformation field in the manner described is to reverse the sign of the shear stress on the plane

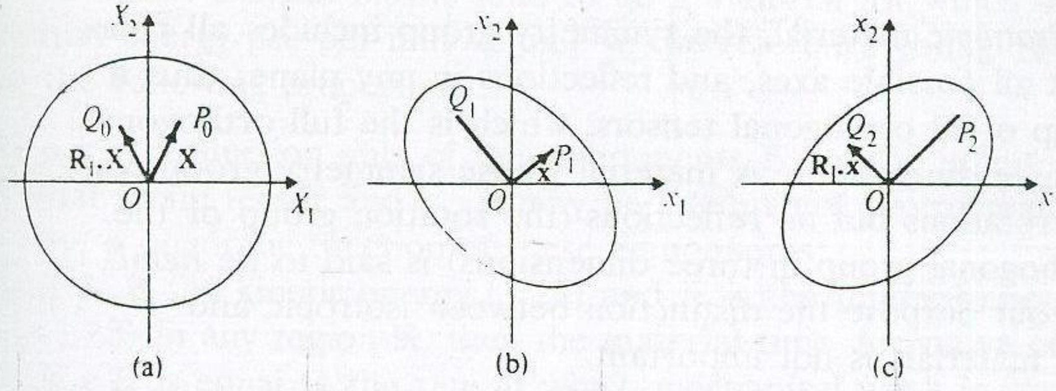

*Figure 8.2* Reflectional symmetry

$x_1 = 0$, we say that the material has reflectional symmetry with respect to this plane, relative to the chosen reference configuration. If the material has this symmetry and the deformation (8.1) gives rise to the stress $\boldsymbol{T}$, then the deformation (8.4) gives rise to the stress $\boldsymbol{R}_1^{\mathrm{T}} \cdot \boldsymbol{T} \cdot \boldsymbol{R}_1$ (the transposition of $\boldsymbol{R}_1$ in the first factor is redundant, because $\boldsymbol{R}_1$ is symmetric, but is introduced for consistency with the corresponding result for rotational symmetries).

More generally, a reflection in the plane through $O$ normal to a unit vector $\boldsymbol{n}$ is defined by a tensor $\boldsymbol{R}$ with components $R_{ij}$, where

$$\boldsymbol{R} = \boldsymbol{I} - 2\boldsymbol{n} \otimes \boldsymbol{n}, \qquad R_{ij} = \delta_{ij} - 2n_i n_j$$

It is easily verified that $\boldsymbol{R}$ is a symmetric improper orthogonal tensor (that is an orthogonal tensor with determinant equal to $-1$). A material has reflectional symmetry for reflections in the planes normal to $\boldsymbol{n}$ if the deformation

$$\boldsymbol{R} \cdot \boldsymbol{x} = \boldsymbol{F} \cdot \boldsymbol{R} \cdot \boldsymbol{X} \tag{8.6}$$

gives rise to the stress $\boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{T} \cdot \boldsymbol{R}$ when the deformation (8.1) gives rise to the stress $\boldsymbol{T}$.

Reflectional symmetry with respect to planes normal to the $X_1$-axis means that the tangential force required to produce a simple shear in (say) the positive $X_2$ direction on the planes $X_1 =$ constant, is equal in magnitude but opposite in direction to that required to produce a shear of the same magnitude in the negative $X_2$ direction on the same planes.

*Symmetry groups.* The set of tensors, such as the rotation tensors $\boldsymbol{Q}$ and the reflection tensors $\boldsymbol{R}$, which define the symmetry properties of a material, form a group (in the technical algebraic sense of the term) which is called the *symmetry group* of the material.

For an *isotropic* material, the symmetry group includes all rotations about all possible axes, and reflections in any plane; thus it is the group of all orthogonal tensors, which is the full orthogonal group in three dimensions. A material whose symmetry group consists of all rotations but no reflections (the rotation group or the proper orthogonal group in three dimensions) is said to be *hemitropic*. For our purpose the distinction between isotropic and hemitropic materials is not important.

Materials which have fewer material symmetries than an isotropic material are said to be *anisotropic*. The symmetry group for an anisotropic material is a subgroup of the full orthogonal group.

A material whose symmetry group includes all rotations about a specified axis is said to be *transversely isotropic* about that axis. Various reflectional symmetries may or may not be added; again the distinctions are not important here.

A material which has reflectional symmetry with respect to each of three mutually orthogonal planes is said to be *orthotropic*. To a good approximation, wood is an example of such a material.

The symmetry group for an orthotropic material is a finite group, composed of the unit tensor, three reflection tensors, and their inner products. Other finite subgroups of the full orthogonal group in three dimensions are symmetry groups for materials with various kinds of *crystal symmetry*. The rotations which occur in these symmetry groups are rotations through multiples of $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. Accounts of the crystallographic groups can be found in texts on crystallography.

For the most part we shall concentrate on isotropy, which is the simplest and most important case, and make only occasional references to anisotropic materials.

## 8.3 Linear elasticity

Many solid materials, and especially the common engineering materials such as metals, concrete, wood, etc., have the property that they only undergo very small changes of shape when they are subjected to the forces which they normally encounter. They also have a natural shape to which they will return if forces are applied to them and then removed (provided that the forces are not too large). The theory of linear elasticity provides an excellent model of the mechanical behaviour of such materials.

We define a linear elastic solid to be a material for which the internal energy $\rho_0 e$ per unit volume in the reference configuration has the following properties:

(a) $\rho_0 e$ is a function only of the components $E_{ij}$ of the infinitesimal strain tensor and is, or may be adequately approximated by, a quadratic function of these components;
(b) if $K$ is the kinetic energy (7.27) and $E$ is the internal energy (7.28) in any region $\mathcal{R}$, then the material time derivative of $K+E$ is equal to the rate at which mechanical work is done by the surface and body forces acting on $\mathcal{R}$.

It is conventional to denote $\rho_0 e$ by $W$, and to call $W$ the strain–energy function. Thus (a) states that $W$ has the form

$$W = \tfrac{1}{2} C_{ijkl} E_{ij} E_{kl} \tag{8.7}$$

where $C_{ijkl}$ are constants. Property (b) is a restatement of the law of conservation of energy (Section 7.6) with heat flux assumed to be absent, or neglected. Properties (a) and (b) together state that all the mechanical work done on $\mathcal{R}$ either creates kinetic energy, or is stored as potential energy (which is called the *strain energy*) which depends only on the deformation. The system is conservative; in a closed cycle of deformation the strain energy is stored and then released so that no net work is done on the body.

The more general case in which $W$ is allowed to depend also on temperature or entropy, and in which heat flux is permitted, leads to the theory of *linear thermoelasticity*. We shall not develop this theory.

It should be noted at the outset that a constitutive equation based on (8.7) will necessarily fail to satisfy one of the requirements stated in Section 8.1 for, as was shown in Section 6.6, the components $E_{ij}$ do not remain constant in a finite rotation, and so $W$ as defined by (8.7) must change when a body rotates without change of shape. This is not reasonable physically. However, if attention is restricted to motions in which the rotation is small, then the change in $E_{ij}$ is of second order in the rotation components. The theory of linear elasticity is essentially an approximate theory which is valid for values of $E_{ij}$ and $\Omega_{ij}$ which are small compared to one. The theory is nevertheless very useful because the approximation is an excellent one in many applications. It is consistent with the approximation involved in adopting (8.7) to neglect $E_{ij}$ compared to one, and this will be done whenever it is convenient to do so.

Suppose we change from a coordinate system with base vectors $\boldsymbol{e}_i$ to a new coordinate system with base vectors $\bar{\boldsymbol{e}}_i$, such that

$$\bar{\boldsymbol{e}}_i = M_{ij}\boldsymbol{e}_j$$

and $(M_{ij})$ is an orthogonal matrix. Then the infinitesimal strain components $E_{ij}$ and $\bar{E}_{ij}$ in the old and new systems are related by the usual tensor transformation rule

$$\bar{E}_{rs} = M_{ri}M_{sj}E_{ij}, \qquad E_{ij} = M_{ri}M_{sj}\bar{E}_{rs} \tag{8.8}$$

The strain energy $W$ can also be expressed as a quadratic function of the components $\bar{E}_{ij}$, as

$$W = \tfrac{1}{2}\bar{C}_{ijkl}\bar{E}_{ij}\bar{E}_{kl} \tag{8.9}$$

However $W$ is a scalar, which is not affected by a change of coordinate system, and so the expressions (8.7) and (8.9) are the same. Hence, using (8.8),

$$\bar{C}_{pqrs}\bar{E}_{pq}\bar{E}_{rs} = C_{ijkl}E_{ij}E_{kl} = C_{ijkl}M_{pi}M_{qj}M_{rk}M_{sl}\bar{E}_{pq}\bar{E}_{rs}$$

This is an identity for all values of $\bar{E}_{ij}$, and so

$$\bar{C}_{pqrs} = M_{pi}M_{qj}M_{rk}M_{sl}C_{ijkl}$$

Hence $C_{ijkl}$ are components of a fourth-order tensor.

The $3^4 = 81$ constants $C_{ijkl}$ are called *elastic constants.* They have the dimensions of stress and their values characterize particular linear elastic materials. The elastic constants are not all independent. By interchanging the dummy indices $i$ and $j$ in (8.7), we obtain

$$W = \tfrac{1}{2}C_{jikl}E_{ji}E_{kl}$$

However, $E_{ij} = E_{ji}$, and so

$$W = \tfrac{1}{2}C_{jikl}E_{ij}E_{kl} = \tfrac{1}{2}\{\tfrac{1}{2}(C_{ijkl} + C_{jikl})\}E_{ij}E_{kl}$$

Thus $C_{ijkl}$ may be replaced by $\tfrac{1}{2}(C_{ijkl} + C_{jikl})$, which is symmetric with respect to interchanges of $i$ and $j$. Hence, without loss of generality, $C_{ijkl}$ may be assumed to be symmetric with respect to interchanges of its first two indices. Similarly, $C_{ijkl}$ may be assumed to be symmetric with respect to interchanges of its third and fourth indices. Thus

$$C_{ijkl} = C_{jikl} = C_{ijlk} \qquad (i, j, k, l = 1, 2, 3) \tag{8.10}$$

The symmetries (8.10) reduce the number of independent elastic constants to 36. Furthermore, by simultaneously interchanging the indices $i$ and $k$ and the indices $j$ and $l$, there follows

$$W=\tfrac{1}{2}C_{klij}E_{ij}E_{kl}=\tfrac{1}{2}\{\tfrac{1}{2}(C_{ijkl}+C_{klij})\}E_{ij}E_{kl}$$

Hence no generality is lost by assuming that $C_{ijkl}$ also has the index symmetries

$$C_{ijkl}=C_{klij} \tag{8.11}$$

The symmetries (8.11) further reduce the number of independent elastic constants to 21.

A further requirement on $W$ is that the stored elastic energy must be positive, so that (8.7) is a positive definite quadratic form in the $E_{ij}$.

Any material symmetry further reduces the number of independent elastic constants. We return to this point below.

So far, property (b) of linear elastic solids has not been employed. From (7.31), with $e$ replaced by $W/\rho_0$, and the heat flux terms neglected, we have

$$T_{ij}\frac{\partial v_i}{\partial x_j}=\frac{\rho}{\rho_0}\frac{\mathrm{D}W}{\mathrm{D}t} \tag{8.12}$$

Since, by (7.7) and (7.8), $\rho/\rho_0=1+O(E_{ij})$, to the order of approximation used in small-deformation theory we may replace $\rho$ by $\rho_0$, and write

$$T_{ij}\frac{\partial v_i}{\partial x_j}=\frac{\mathrm{D}W}{\mathrm{D}t}$$

It was shown in Section 7.6 that $T_{ij}\,\partial v_i/\partial x_j=T_{ij}D_{ij}$, and so

$$T_{ij}D_{ij}=\frac{\mathrm{D}W}{\mathrm{D}t} \tag{8.13}$$

Now, since $W$ depends only on $E_{ij}$, (8.13) gives

$$T_{ij}D_{ij}=\frac{\partial W}{\partial E_{ij}}\frac{\mathrm{D}E_{ij}}{\mathrm{D}t}$$

and (6.77) then gives, to the required order of approximation,

$$T_{ij}D_{ij}=\frac{\partial W}{\partial E_{ij}}D_{ij}$$

This is an identity which holds for all values of $D_{ij}$, and so

$$T_{ij}=\frac{\partial W}{\partial E_{ij}}$$

However, from (8.7) and (8.11),

$$\begin{aligned}\frac{\partial W}{\partial E_{ij}}&=\frac{1}{2}\frac{\partial}{\partial E_{ij}}(C_{pqrs}E_{pq}E_{rs})\\&=\tfrac{1}{2}C_{pqrs}(\delta_{ip}\delta_{jq}E_{rs}+\delta_{ir}\delta_{js}E_{pq})\\&=\tfrac{1}{2}(C_{ijrs}E_{rs}+C_{pqij}E_{pq})\\&=C_{ijrs}E_{rs}\end{aligned}$$

Hence

$$T_{ij}=C_{ijrs}E_{rs} \tag{8.14}$$

and this is the constitutive equation for a linear elastic solid. It is evident that the stress components are linear functions of the infinitesimal strain components.

An alternative formulation of linear elasticity theory is based on the assumption that the stress components $T_{ij}$ are (or can adequately be approximated by) linear functions of the infinitesimal strain components $E_{ij}$, so that (8.14) is taken as the starting point rather than as a consequence of (8.7). In such a formulation there is no loss of generality in giving $C_{ijkl}$ the index symmetries (8.10), but (8.11) does not obtain unless further assumptions are made. A material with constitutive equation (8.14) but lacking the index symmetry (8.11) has the unrealistic property that work can be extracted from it in a closed cycle of deformation. We therefore prefer to base the theory on (8.7), from which (8.11) follows automatically.

The number of independent elastic constants is further reduced if the material possesses any material symmetry. Suppose for example that the material has the reflectional symmetry with respect to the $(X_2, X_3)$ planes which is associated with the tensor $\boldsymbol{R}_1$ which is defined by (8.5). Since $E_{ij}=\frac{1}{2}(F_{ij}+F_{ji})-\delta_{ij}$, it is easily seen that the effect of replacing the deformation (8.1) by the deformation (8.3) is to replace $E_{12}$ by $-E_{12}$, and $E_{13}$ by $-E_{13}$, while leaving the other components $E_{ij}$ unaltered. However, if $\mathscr{R}_1$ belongs to the symmetry group, $W$ must be unchanged by this substitution. Hence, if the material has this symmetry, then

$$W(E_{11},E_{22},E_{33},E_{23},E_{31},E_{12})=W(E_{11},E_{22},E_{33},E_{23},-E_{31},-E_{12}) \tag{8.15}$$

and this relation must hold identically for all $E_{ij}$. By writing (8.7) in full with the above two sets of arguments, or by considering special cases, it follows from (8.7) and (8.15) that

$$C_{1112}=C_{1113}=C_{1222}=C_{1223}=C_{1233}=C_{1322}=C_{1323}=C_{1333}=0$$

Other material symmetries impose further restrictions on the elastic constants. The various possibilities are described in texts on linear elasticity. We omit the details and proceed to the case of isotropic materials.

The symmetry group for isotropic materials includes all proper orthogonal tensors $\boldsymbol{Q}$. Suppose, as before, that $E_{ij}$ are the components of infinitesimal strain which correspond to the deformation (8.1). Then the corresponding stress components $T_{ij}$ are given by (8.14). The infinitesimal strain components which correspond to the deformation (8.2) are

$$\bar{E}_{pq}=Q_{kp}(\tfrac{1}{2}F_{kl}+\tfrac{1}{2}F_{lk}-\delta_{kl})Q_{lq}=Q_{kp}Q_{lq}E_{kl} \tag{8.16}$$

and the associated stress components are

$$\bar{T}_{rs}=C_{rspq}\bar{E}_{pq} \tag{8.17}$$

Now if $\boldsymbol{Q}$ belongs to the symmetry group, then

$$T_{ij}=Q_{ir}Q_{js}\bar{T}_{rs} \tag{8.18}$$

and hence, from (8.16), (8.17) and (8.18),

$$T_{ij}=Q_{ir}Q_{js}Q_{kp}Q_{lq}C_{rspq}E_{kl} \tag{8.19}$$

It follows, by comparing (8.14) and (8.19), that

$$C_{ijkl}=Q_{ir}Q_{js}Q_{kp}Q_{lq}C_{rspq} \tag{8.20}$$

and, if the material is isotropic, this must hold for all orthogonal tensors $\boldsymbol{Q}$. However, (8.20) then becomes a statement that $C_{ijkl}$ are components of a fourth-order *isotropic* tensor (Section 3.5). The most general fourth-order isotropic tensor is given by (3.37). Hence $C_{ijkl}$ take the form

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\nu\delta_{il}\delta_{jk} \tag{8.21}$$

and the constitutive equation (8.14) becomes

$$T_{ij}=\lambda\delta_{ij}E_{kk}+\mu E_{ij}+\nu E_{ji}$$

Since $E_{ij}=E_{ji}$, no generality is lost by setting $\nu=\mu$, so that

$$T_{ij}=\lambda\delta_{ij}E_{kk}+2\mu E_{ij} \tag{8.22}$$

or, equivalently, in tensor notation

$$\boldsymbol{T}=\lambda \boldsymbol{I} \operatorname{tr} \boldsymbol{E}+2\mu \boldsymbol{E}$$

Equation (8.22) is the constitutive equation for an isotropic linear elastic solid; such a material is characterized by the two elastic constants $\lambda$ and $\mu$.

We observe that the form (8.21) possesses the index symmetry $C_{ijkl}=C_{klij}$. Thus for an *isotropic* material we arrive at (8.22) regardless of whether we adopt (8.7) or (8.14) as the starting point.

## 8.4 Newtonian viscous fluids

In experiments on water, air and many other fluids, it is observed that in a simple shearing flow (Section 6.10) the shearing stress on the shear planes is proportional to the shear rate $s$, to an extremely good approximation and over a very wide range of shear rates. This behaviour is characteristic of a *Newtonian viscous fluid* or a *linear viscous fluid*. This model of fluid behaviour describes the mechanical properties of many fluids, including the commonest fluids, air and water, very well indeed.

We consider fluids with constitutive equations of the form

$$T_{ij}=-p(\rho, \theta)\delta_{ij}+B_{ijkl}(\rho, \theta)D_{kl} \tag{8.23}$$

where $\theta$ is the temperature. In a fluid at rest, $D_{kl}=0$, and (8.23) reduces to

$$T_{ij}=-p(\rho, \theta)\delta_{ij} \tag{8.24}$$

which is the constitutive equation employed in hydrostatics, with $p(\rho, \theta)$ representing the hydrostatic pressure. Thus (8.23) specifies that in a fluid in motion the additional stress over the hydrostatic pressure is linear in the components of the rate of deformation tensor.

If the fluid is isotropic, then arguments similar to those used in Section 8.3 to reduce (8.14) to (8.22) lead to the conclusion that $B_{ijkl}$ are (like $C_{ijkl}$ for an isotropic linear elastic solid) the components of a fourth-order isotropic tensor, and then (8.23) takes the form

$$T_{ij}=\{-p(\rho, \theta)+\lambda(\rho, \theta)D_{kk}\}\delta_{ij}+2\mu(\rho, \theta)D_{ij} \tag{8.25}$$

or, equivalently

$$\boldsymbol{T}=\{-p(\rho, \theta)+\lambda(\rho, \theta)\operatorname{tr} \boldsymbol{D}\}\boldsymbol{I}+2\mu(\rho, \theta)\boldsymbol{D}$$

Here the viscosity coefficients $\lambda(\rho, \theta)$ and $\mu(\rho, \theta)$ are of course, not the same as the elastic constants $\lambda$ and $\mu$ which were introduced in Section 8.3. A particular linear viscous fluid is characterized by the two coefficients $\lambda$ and $\mu$.

It was shown in Section 6.9 that $D_{ij} = 0$ in a rigid-body motion and that the superposition of a rigid-body motion on a given motion does not change the value of $D_{ij}$. Hence the right-hand side of (8.25) is not affected by a superimposed rigid-body motion. Therefore the constitutive equation (8.25) has the required property of being independent of superimposed rigid-body motions. This is in contrast to the constitutive equation of linear elasticity theory, which, it was emphasized in Section 8.3, is necessarily an approximate theory and is valid only for small rotations and deformations. Equation (8.25) is a possible exact constitutive equation for a viscous fluid. In practice, it is found that (8.25) serves extremely well to describe the mechanical behaviour of many fluids.

In fluid mechanics texts it is usual to assume, as we have done here, that the fluid is isotropic. In fact it can be shown that isotropy is a consequence of (8.23) and the requirement that the stress is not affected by rigid-body motions, and so isotropy need not be introduced as a separate assumption. We shall demonstrate this, in a more general context, in Section 10.3. It does not follow that all fluids are necessarily isotropic. Fluids with anisotropic properties do exist, but they require more general constitutive equations than (8.23) for their description.

Several special cases of (8.25) are of interest. If the stress is a hydrostatic pressure (see Section 5.9) then

$$T_{ij} = \tfrac{1}{3} T_{kk} \delta_{ij} = \{-p(\rho, \theta) + (\lambda + \tfrac{2}{3}\mu) D_{kk}\} \delta_{ij}$$

It is often assumed that in such a state of pure hydrostatic stress, the stress depends only on $\rho$ and $\theta$ and not on the dilatation rate $D_{kk}$. If this is the case then $\lambda + \tfrac{2}{3}\mu = 0$, and this relation is often adopted.

If the material is *inviscid*, then $\lambda = 0$ and $\mu = 0$, and the constitutive equation reduces to (8.24). The stress in an inviscid fluid is always hydrostatic.

If the fluid is *incompressible*, then $\rho$ is constant and $D_{kk} = 0$. Incompressibility is a kinematic constraint which gives rise to a reaction stress. The reaction to incompressibility is an arbitrary hydrostatic pressure which can be superimposed on the stress field without causing any deformation; this pressure does no work in any deformation which satisfies the incompressibility constraint. Such a hydrostatic pressure is not determined by constitutive equations but can only be found through the equations of motion or of equilibrium, and the boundary conditions. Thus for an incompressible viscous fluid, (8.25) reduces to

$$T_{ij} = -p\delta_{ij} + 2\mu(\theta)D_{ij}, \qquad \text{or} \qquad \boldsymbol{T} = -p\boldsymbol{I} + 2\mu(\theta)\boldsymbol{D} \tag{8.26}$$

where $p$ is arbitrary, $\mu$ depends only on $\theta$, and the term $\lambda D_{kk}$ has been absorbed into the arbitrary function $p$. We note that in the limit as the material becomes incompressible, $D_{kk} \to 0$ and $\lambda \to \infty$ in such a way that $\lambda D_{kk}$ tends to a finite limit.

If the fluid is both inviscid and incompressible (such a fluid is called an *ideal fluid*) then

$$T_{ij} = -p\delta_{ij}, \qquad \text{or} \qquad \boldsymbol{T} = -p\boldsymbol{I} \tag{8.27}$$

where $p$ is arbitrary in the sense that it is not determined by a constitutive equation.

## 8.5 Linear viscoelasticity

Many materials (especially materials which are usually described as 'plastics') possess both some of the characteristics of elastic solids and some of the characteristics of viscous fluids. Such materials are termed *viscoelastic*. The phenomenon of viscoelasticity is illustrated by *creep* and *stress-relaxation* experiments. For simplicity, consider the case of simple tension. Suppose a tension $F_0$ is rapidly applied to an initially stress-free viscoelastic string at time $t = 0$, and then held, constant, as illustrated in Fig. 8.3(a). The

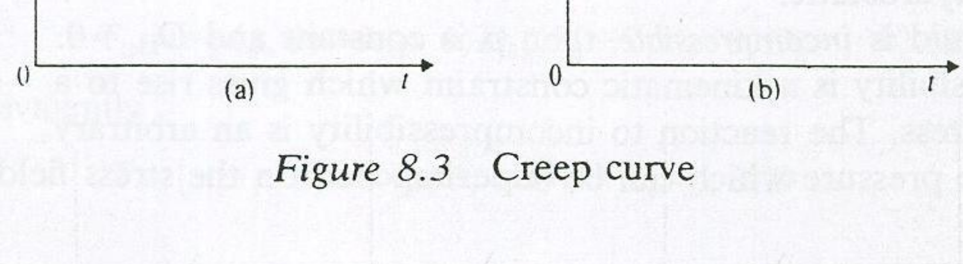

*Figure 8.3* Creep curve

corresponding relation between the elongation $e$ and time $t$ may be of the form shown in Fig. 8.3(b), with an initial elongation $e_0$ (such as would occur in an elastic material) followed by an increasing elongation under the maintained load. This illustrates the phenomenon of creep. If the material is a viscoelastic *solid*, the elongation tends to a finite limit $e_\infty$ as $t \to \infty$; if the material is a viscoelastic *fluid*, the elongation continues indefinitely.

Alternatively, suppose that at $t=0$ the string is given an elongation $e_0$ and held in this position (Fig. 8.4(a)). The resulting force response is shown in Fig. 8.4(b); the force rises instantaneously to $F_0$ at $t=0$ and then decays. This is stress relaxation. For a fluid, $F \to 0$ as $t \to \infty$; in a solid, $F$ tends to a finite limit $F_\infty$ as $t \to \infty$.

We consider here only infinitesimal deformations, so that the use of the infinitesimal strain tensor is appropriate. With the behaviour illustrated in Fig. 8.4 as motivation, we assume that an increment $\delta E_{ij}$ in the strain components at time $\tau$ gives rise to increments $\delta T_{ij}$ in the stress components at subsequent times $t$, the magnitude of these increments depending on the lapse of time since the strain increment was applied. Thus

$$\delta T_{ij}(t) = G_{ijkl}(t-\tau)\, \delta E_{kl}(\tau) \tag{8.28}$$

where we expect $G_{ijkl}$ to be decreasing functions of $t-\tau$. The *superposition principle* is also assumed, according to which the total stress at time $t$ is obtained by superimposing the effect at time $t$ of all the strain increments at times $\tau < t$. Thus

$$T_{ij}(t) = \int_{-\infty}^{t} G_{ijkl}(t-\tau)\, \frac{\mathrm{d}E_{kl}(\tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau \tag{8.29}$$

This is the constitutive equation for linear viscoelasticity. The functions $G_{ijkl}$ are called *relaxation functions*. If the strain was

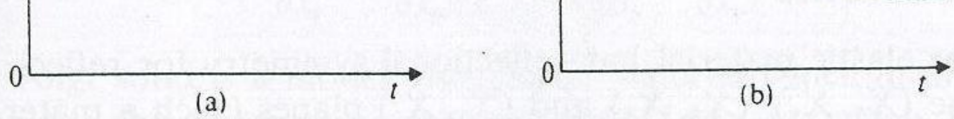

*Figure 8.4* Stress-relaxation curve

zero in the remote past, so that $E_{kl} \to 0$ as $\tau \to -\infty$, (8.29) can be expressed in an alternative form by carrying out an integration by parts, as follows:

$$T_{ij}(t) = E_{kl}(t)G_{ijkl}(0) - \int_{-\infty}^{t} E_{kl} \frac{\mathrm{d}}{\mathrm{d}\tau}\{G_{ijkl}(t-\tau)\}\,\mathrm{d}\tau \qquad (8.30)$$

The stress-relaxation functions $G_{ijkl}(t-\tau)$ have the index symmetries $G_{ijkl} = G_{jikl} = G_{ijlk}$, but not the index symmetry $G_{ijkl} = G_{klij}$, unless this is introduced as a further assumption. If the material is isotropic, then $G_{ijkl}$ are components of a fourth-order isotropic tensor, and, for example, (8.29) reduces to

$$T_{ij} = \delta_{ij} \int_{-\infty}^{t} \lambda(t-\tau) \frac{\mathrm{d}E_{kk}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau + 2\int_{-\infty}^{t} \mu(t-\tau)\frac{\mathrm{d}E_{ij}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \qquad (8.31)$$

and only two relaxation functions $\lambda(t-\tau)$ and $\mu(t-\tau)$ are required to describe the material.

The inverse relation to (8.29) is

$$E_{ij}(\tau) = \int_{-\infty}^{t} J_{ijkl}(t-\tau)\frac{\mathrm{d}T_{kl}}{\mathrm{d}\tau}\,\mathrm{d}\tau \qquad (8.32)$$

The functions $J_{ijkl}(t-\tau)$ are known as *creep functions*; they have the same index symmetries as $G_{ijkl}(t-\tau)$ and are components of a fourth-order isotropic tensor in the case in which the material is isotropic.

Linear viscoelasticity has the same limitations as linear elasticity: it is necessarily an approximate theory which can only be applicable when the strain and rotation components are small.

In a sense, linear elasticity can be regarded as the limiting case of linear viscoelasticity in which the relaxation functions are independent of $t$; and a Newtonian viscous fluid as the limiting case of an isotropic linear viscoelastic material in which the relaxation functions $\lambda(t-\tau)$ and $\mu(t-\tau)$ take the forms $\lambda\delta(t-\tau)$ and $\mu\delta(t-\tau)$ respectively, where $\lambda$ and $\mu$ are the viscosity coefficients and $\delta(t-\tau)$ is the Dirac delta function.

## 8.6 Problems

1. A linear elastic material has reflectional symmetry for reflections in the $(X_2, X_3)$, $(X_3, X_1)$ and $(X_1, X_2)$ planes (such a material is said to be *orthotropic*). Show that it has nine independent elastic constants.

2. Show that a transversely isotropic linear elastic solid has five independent elastic constants, and find the form of $W$ for a linear elastic solid which is transversely isotropic with respect to the $X_3$-axis.

3. From the constitutive equation (8.22) and the equation of motion (7.22), with $\boldsymbol{b}=\mathbf{0}$, derive Navier's equations for an isotropic linear elastic solid:

$$(\lambda+\mu)\frac{\partial^2 u_k}{\partial x_k\,\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_k^2}=\rho\frac{\partial^2 u_i}{\partial t^2}$$

4. In simple tension of an isotropic linear elastic solid. $T_{11}=EE_{11}$, $T_{22}=T_{33}=T_{23}=T_{31}=T_{12}=0$, and $E_{22}=E_{33}=-\nu E_{11}$, where $E$ is Young's modulus and $\nu$ is Poisson's ratio. Prove that $E=\mu(3\lambda+2\mu)/(\lambda+\mu)$ and $\nu=\frac{1}{2}\lambda/(\lambda+\mu)$. Show that the constitutive equation (8.22) can be expressed in the form

$$\boldsymbol{E}=\frac{1}{E}\{(1+\nu)\boldsymbol{T}-\nu\boldsymbol{I}\operatorname{tr}\boldsymbol{T}\}$$

5. Prove that necessary and sufficient conditions for $W$ to be positive definite for an isotropic linear elastic solid are $\mu>0$, $\lambda+\frac{2}{3}\mu>0$.

6. In plane stress or in plane strain, the equilibrium equations reduce to (5.42). Show that these equations are identically satisfied if the stress components are expressed in terms of Airy's stress function $\chi$, as $T_{11}=\partial^2\chi/\partial x_2^2$, $T_{22}=\partial^2\chi/\partial x_1^2$, $T_{12}=-\partial^2\chi/\partial x_1\,\partial x_2$. Prove that in plane stress or plane strain of an isotropic linear elastic solid, $\chi$ satisfies the biharmonic equation

$$\nabla^4\chi=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)^2\chi=0$$

7. From the constitutive equation (8.26) and the equations of motion (7.22) derive the Navier–Stokes equations for an incompressible Newtonian fluid

$$\rho b_i-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j\,\partial x_j}=\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)$$

8. A Voigt solid is a model viscoelastic material which in uniaxial tension has the stress–strain relation $\sigma=E_0(\varepsilon+t_0\dot{\varepsilon})$, where $E_0$ and $t_0$ are constants. Sketch the creep and stress-relaxation curves

for this material. Show that the relaxation function is $E_0\{1+t_0\,\delta(t-\tau)\}$. Give a three-dimensional generalization of the above constitutive equation for an incompressible isotropic material.

9. A Maxwell fluid is a model viscoelastic material which in uniaxial tension has the stress–strain relation $E_1\dot{\varepsilon}=\dot{\sigma}+\sigma/t_1$. Sketch the creep and stress relaxation curves. Show that the stress relaxation function is $E_1\exp\{-(t-\tau)/t_1\}$. Hence give a three-dimensional generalization for an isotropic incompressible material in the integral form (8.31).